

Stability and Rate of Convergence of the Steiner Symmetrization

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Abstract

We present a direct analytic method towards an estimate for the rate of convergence (to the Euclidean Ball) of Steiner symmetrizations. To this end we present a modified version of a known stability property of the Steiner symmetrization.

1 Introduction and results

Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be some fixed Euclidean structure, and let \mathcal{K}^n be the class of all compact convex sets in \mathbb{R}^n . Denote by D_n the Euclidean unit ball, by S^{n-1} its boundary and by $\kappa_n = |D_n|$ its Lebesgue measure. Fix a direction $u \in S^{n-1}$ and denote its orthogonal hyperplane by $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$. Obviously, each point $x \in \mathbb{R}^n$ can be uniquely decomposed as $x = y + tu$ where $y \in H$ and $t \in \mathbb{R}$. The Steiner symmetral of a set K with respect to u is defined to be

$$S_u(K) = \left\{ (y, t) : K \cap (y + \mathbb{R}u) \neq \emptyset, \quad |t| \leq \frac{|K \cap (y + \mathbb{R}u)|}{2} \right\}.$$

The Steiner symmetrization has several important properties. For one, it reduces the surface area while preserving volume. Clearly, this process makes the set more “round” in some sense, so one would expect that applying multiple Steiner symmetrizations is a process that converges to the Euclidean ball - the only fixed point of this operation. It was shown by Gross [4] that for each convex set there exists a sequence of symmetrizations that converges in the Hausdorff metric to a ball with the same volume. This result was improved by Mani-Levitska [9] where it was shown that a random sequence of Steiner symmetrizations applied to a convex set, converges almost surely to a ball. However, these proofs do not provide results regarding the rate of convergence. The first estimate of the rate is due to Hadwiger [5], who showed that $\left(c \frac{\sqrt{n}}{\varepsilon^2}\right)^n$ symmetrizations are enough to transform a convex set to a new set with Hausdorff distance at most ε from the Euclidean ball. Later, Bourgain, Lindenstrauss

and Milman [2] proved an isomorphic result, stating that in order to reach some fixed distance from the Euclidean Ball, roughly $n \log n$ symmetrizations suffice. In recent years this bound was reduced to $3n$ by Klartag and Milman [7]. Klartag [8] also improved the isometric result of Hadwiger, showing that the rate of convergence is almost exponential. More precisely:

Theorem 1.1 (Klartag). *Let $K \in \mathcal{K}^n$ be a convex body with $|K| = |D_n|$, and let $\varepsilon \in (0, \frac{1}{2})$. There exist $Cn^4(\log \varepsilon)^2$ Steiner symmetrization transforming K into a body K' satisfying*

$$(1 - \varepsilon)D_n \subset K \subset (1 + \varepsilon)D_n.$$

In [8] Klartag first provided a bound of $Cn|\log \varepsilon|$ steps on the convergence rate when applying the *Minkowski symmetrization* $M_u K$, a linear operation on the support function, by means of controlling the decay of the non-constant spherical harmonics of the support function. The proof of Theorem 1.1 consists mainly of the bound for Minkowski symmetrizations, together with the inclusion $S_u K \subseteq M_u K$. A byproduct of this approximation is that the bound for Steiner symmetrizations is polynomial in the dimension n rather than linear. It is conjectured that the correct dependence is indeed linear, as in the case of Minkowski symmetrizations. The goal of this paper is to provide a direct estimate for the convergence rate of Steiner symmetrizations in the Nikodym pseudo metric, defined in Section 3. It may be formulated as follows, where $A \Delta B$ is the *symmetric difference* of the sets A and B .

Theorem 1.2. *Let $K \in \mathcal{K}^n$ be a convex body with $|K| = |D_n|$ and let $\varepsilon \in (0, 1)$. There exist $c \frac{n^{13} \log^3 n}{\varepsilon^\gamma}$ Steiner symmetrizations transforming K into a body K' satisfying*

$$\frac{|K' \Delta D_n|}{|D_n|} < \varepsilon,$$

where $\gamma = 4 + \frac{2}{\log n}$.

Obviously, Theorem 1.2 provides a non optimal bound (for example, by equivalence of the Hausdorff and Nikodym metrics, one can derive a better bound from Theorem 1.1). However, the polynomial bound presented in this proof is obtained using a self contained, direct analysis of Steiner symmetrization, which may lead to similar results in the case of non convex sets, where there are no estimates analogous to Theorem 1.1. The main ingredient of our proof is a quantitative estimate regarding the change in surface area under a Steiner symmetrization. It is a well known fact that surface area decreases under a Steiner symmetrization. However, a quantitative version of this statement was only recently provided, by Barchiesi, Cagnetti and Fusco [1]. Their statement contains factors which are exponential in the dimension and have a direct effect on the estimate of the convergence rate. In Section 3 we provide a slightly different version with an improved dependence on the dimension. To

this end we require a Poincaré type inequality for convex domains, which we obtain in the following section.

2 Poincaré type inequalities for convex domains

We wish to establish a weighted Poincaré type inequality for convex domains. We denote by $\rho : K \rightarrow \mathbb{R}^+$ the distance to the boundary of K , that is

$$\rho(x) = \min_{y \in \partial K} \{|x - y|\}.$$

Our main result in this section is the following theorem:

Theorem 2.1. *Let $n \geq 2$ and let $K \in \mathcal{K}^n$ be such that $rD_n \subseteq K \subseteq RD_n$. If $b : K \rightarrow \mathbb{R}$ is a bounded function with mean zero with respect to ρ (i.e. $\int_K b\rho = 0$), then for every $\lambda \in (2, \infty)$ one has*

$$\int_K |b| \leq C \left(\frac{\|b\|_\infty |K|}{\beta} \right)^{1-\frac{1}{\lambda}} \cdot \left(n \frac{R}{r} \int_K |\nabla b| \rho \right)^{\frac{1}{\lambda}},$$

where $\beta = \frac{\lambda-2}{\lambda-1} \in (0, 1)$.

We collect a few technical lemmas before proving Theorem 2.1.

Lemma 2.2. *Let $n \geq 1$ and let $K \in \mathcal{K}^n$ be such that $rD_n \subseteq K \subseteq RD_n$. Then*

$$\frac{n}{R} \leq \frac{|\partial K|}{|K|} \leq \frac{n}{r}.$$

Proof. By the definition we have:

$$|\partial K| = \lim_{t \rightarrow 0} \frac{|K + tD_n| - |K|}{t} \leq \lim_{t \rightarrow 0} \frac{|K + \frac{t}{r}K| - |K|}{t} = \frac{n|K|}{r}.$$

Replacing D_n with K/R in the above limit yields the other direction. \square

Lemma 2.3. *Let $n \geq 1$, $K \in \mathcal{K}^n$. For every $\beta \in (0, 1)$ we have*

$$I_\beta = \int_K \frac{1}{\rho^{1-\beta}} < \frac{Cn^{1-\beta}|K|}{\beta r^{1-\beta}},$$

where r is the inner radius of K , and C is some positive constant.

Proof. First, recall the Beta function defined for positive x and y by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

The function ρ is bounded (from above and below) by the K -distance-to-the-boundary function $\rho_K(x) = \min_{y \in \partial K} \{|x - y|_K\} = 1 - \|x\|_K$, whose corresponding integral is easily estimated. Indeed, if $rD_n \subseteq K \subseteq RD_n$ then

$$\frac{1}{R}|x| \leq \|x\|_K \leq \frac{1}{r}|x|.$$

In particular we get a lower bound on ρ :

$$\rho(x) = \min_{y \in \partial K} |x - y| \geq r \min_{y \in \partial K} \|x - y\|_K = r\rho_K(x). \quad (1)$$

By Fubini's theorem:

$$\begin{aligned} \int_K \frac{1}{\rho_K(x)^{1-\beta}} &= \int_0^\infty \left| \left\{ x \in K : \rho_K^{-(1-\beta)}(x) > t \right\} \right| dt \\ &= \int_0^1 |K| dt + \int_1^\infty \left| \left\{ x \in K : \rho_K^{-(1-\beta)}(x) > t \right\} \right| dt \\ &= |K| \left(1 + \int_1^\infty \left(1 - (1 - t^{-\frac{1}{1-\beta}})^n \right) dt \right) \\ &= |K| \left(1 + \int_0^1 (1 - s^n)(1 - s)^{\beta-2}(1 - \beta) ds \right) \\ &= n|K| \int_0^1 (1 - s)^{\beta-1} s^{n-1} ds = n|K| B(n, \beta) \\ &= n|K| \frac{n + \beta}{\beta} B(n, 1 + \beta) \\ &= n|K| \left(\frac{n + \beta}{\beta} \right) \frac{\Gamma(n)\Gamma(1 + \beta)}{\Gamma(1 + n + \beta)} \\ &< \frac{2n^2|K|\Gamma(n)}{\beta\Gamma(1 + n + \beta)} < \frac{Cn^{1-\beta}|K|}{\beta}, \end{aligned}$$

for some $C > 0$, since $\Gamma(n)n^{1+\beta} < C_1\Gamma(1 + n + \beta)$. Therefore, by (1) we conclude that

$$I_\beta = \int_K \frac{1}{\rho^{1-\beta}} \leq \frac{1}{r^{1-\beta}} \int_K \frac{1}{\rho_K(x)^{1-\beta}} < \frac{Cn^{1-\beta}|K|}{\beta r^{1-\beta}}.$$

□

The last tool we require is the following weighted Poincaré type inequality, due to Chua and Wheeden (in fact, in [3] they prove a more general result).

Theorem 2.4 (Chua, Wheeden). *Let $K \in \mathcal{K}^n$ and let f be a Lipschitz function. If*

$$\int_K f \rho = 0,$$

then

$$\int_K |f| \rho \leq C \operatorname{diam}(K) \int_K |\nabla f| \rho,$$

where $C > 0$ is some universal constant.

We turn now to prove the main result of this section.

Proof of Theorem 2.1. Let $\lambda > 2$. By the Hölder inequality we have

$$\int_K |b|^{\frac{1}{\lambda}} \leq \left(\int_K |b| \rho \right)^{\frac{1}{\lambda}} \cdot \left(\int_K \rho^{\frac{1}{1-\lambda}} \right)^{1-\frac{1}{\lambda}} \quad (2)$$

We write $\lambda = \frac{2-\beta}{1-\beta}$ for $\beta \in (0, 1)$, so that $(\lambda - 1)(1 - \beta) = 1$. By Lemma 2.3,

$$\begin{aligned} \left(\int_K \rho^{\frac{1}{1-\lambda}} \right)^{1-\frac{1}{\lambda}} &= \left(\int_K \frac{1}{\rho^{1-\beta}} \right)^{1-\frac{1}{\lambda}} < \left(\frac{C n^{1-\beta} |K|}{\beta r^{1-\beta}} \right)^{1-\frac{1}{\lambda}} \\ &= \left(\frac{n}{r} \right)^{\frac{1}{\lambda}} \cdot \left(\frac{C |K|}{\beta} \right)^{1-\frac{1}{\lambda}} \end{aligned}$$

Combining the two estimates we get

$$\int_K |b| \leq \|b\|_{\infty}^{1-\frac{1}{\lambda}} \int_K |b|^{\frac{1}{\lambda}} \leq \left(\frac{C \|b\|_{\infty} |K|}{\beta} \right)^{1-\frac{1}{\lambda}} \left(\frac{n}{r} \int_K |b| \rho \right)^{\frac{1}{\lambda}}.$$

Since $\int_K b \rho = 0$, and $\operatorname{diam}(K) \leq 2R$, we may apply Theorem 2.4 to obtain:

$$\int_K |b| \leq \left(\frac{C \|b\|_{\infty} |K|}{\beta} \right)^{1-\frac{1}{\lambda}} \left(\frac{nR}{r} \int_K |\nabla b| \rho \right)^{\frac{1}{\lambda}}.$$

□

3 Stability for the Steiner symmetrization

Let $K \in \mathcal{K}^n$ and $u \in S^{n-1}$. It is well known that the surface area $|\partial K|$ decreases under a Steiner symmetrization, but until recently this phenomenon was not quantified. Barchiesi, Cagnetti and Fusco showed in [1] that for a convex body K satisfying $rD_n \subseteq K \subseteq RD_n$, the following holds

$$A(K, S_u K) \leq n 4^{n+1} \left(\frac{R}{r} \right)^{2n} \sqrt{\delta_u(K)}, \quad (3)$$

where the *surface area deficit* δ_u is defined by

$$\delta_u(K) = 1 - \frac{|\partial S_u K|}{|\partial K|},$$

and the *Nikodym pseudo metric* A is defined by

$$A(K, T) = \inf_{x_0 \in \mathbb{R}^n} \frac{|(rK)\Delta(x_0 + T))|}{|T|},$$

for $r^n = \frac{|T|}{|K|}$. In this section we show that the dependence on the dimension and the quantity $\frac{R}{r}$ in (3) can be reduced to polynomial at the cost of slightly worsening the exponent of $\delta_u(K)$ (i.e. decreasing it below $1/2$). More precisely:

Theorem 3.1. *Let $n \geq 2$ and let $K \in \mathcal{K}^n$ such that $rD_n \subseteq K \subseteq RD_n$. Then for every $\lambda \in (2, \infty)$ and $u \in S^{n-1}$ we have:*

$$A(K, S_u K) \leq C \left(\frac{1}{\beta} \right)^{1-\frac{1}{\lambda}} \left(n \frac{R}{r} \right)^{1+\frac{1}{\lambda}} \delta_u(K)^{\frac{1}{2\lambda}},$$

where $\beta = \frac{\lambda-2}{\lambda-1} \in (0, 1)$, and C is some constant.

The proof of Theorem 3.1 follows the methods of [1], combined with Theorem 2.1. Denote the orthogonal projection of K to u^\perp by $P = Proj_{u^\perp}(K)$. For each $x \in P$, we consider the “fiber above x in K ”, namely $K \cap (x + \mathbb{R}u)$. We denote its length by $L(x) = |K \cap (x + \mathbb{R}u)|$ and its barycenter by $b(x)$. By Brunn’s principle, L is concave. The following lemma gives a local upper bound for the gradient of a concave function, in terms of the distance from the boundary of its domain.

Lemma 3.2. *Let $P \in \mathcal{K}^n$, and denote by $\rho(x) = \text{dist}(x, \partial P)$ the distance to ∂P . If $L : P \rightarrow \mathbb{R}$ is a concave function with oscillation $\Delta L := \sup\{L\} - \inf\{L\}$, then*

$$|\nabla L(y)| \leq \frac{\Delta L}{\rho(y)}.$$

Proof. Let $y \in P$, and consider $x = y - \rho(y)v \in P$, where $v = \frac{\nabla L(y)}{|\nabla L(y)|}$. Then

$$L(y) - L(x) = \int_0^{\rho(y)} \frac{\partial L}{\partial v}(x + tv) dt \geq \int_0^{\rho(y)} \frac{\partial L}{\partial v}(y) dt = \rho(y) |\nabla L(y)|,$$

by concavity. Therefore $|\nabla L(y)| \leq \frac{\Delta L}{\rho(y)}$, as required. \square

Proof of Theorem 3.1. As before, denote $P = Proj_{u^\perp}(K) = Proj_{u^\perp}(S_u K)$ and $\rho(x) = \text{dist}(x, \partial P)$. The fiber $K \cap (x + \mathbb{R}u)$ has endpoints with heights $b \pm \frac{L}{2}$, thus:

$$\begin{aligned} |\partial K| - |\partial S_u K| &= \int_P \left(\sqrt{1 + \left| \nabla b + \frac{\nabla L}{2} \right|^2} + \sqrt{1 + \left| \nabla b - \frac{\nabla L}{2} \right|^2} - 2\sqrt{1 + \left| \frac{\nabla L}{2} \right|^2} \right) \\ &= \int_P \frac{N}{D} \geq \left(\int_P N^{\frac{1}{2}} \right)^2 \left(\int_P D \right)^{-1}, \end{aligned}$$

where

$$\begin{aligned}
N &= \frac{1}{2} \left(\sqrt{1 + \left| \nabla b + \frac{\nabla L}{2} \right|^2} + \sqrt{1 + \left| \nabla b - \frac{\nabla L}{2} \right|^2} \right)^2 - \frac{1}{2} \left(2 \sqrt{1 + \left| \frac{\nabla L}{2} \right|^2} \right)^2 \\
&= \sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2 \right)^2 - \langle \nabla L, \nabla b \rangle^2} - \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2 \right) \\
&= \frac{4 \left(1 + \frac{1}{4} |\nabla L|^2 \right) |\nabla b|^2 - \langle \nabla b, \nabla L \rangle^2}{\sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2 \right)^2 - \langle \nabla L, \nabla b \rangle^2} + \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2 \right)} \\
&\geq \frac{4 |\nabla b|^2}{\sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2 \right)^2 - \langle \nabla L, \nabla b \rangle^2} + \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2 \right)} \equiv \frac{4 |\nabla b|^2}{Q},
\end{aligned}$$

and

$$D = \frac{1}{2} \left(\sqrt{1 + \left| \nabla b + \frac{1}{2} \nabla L \right|^2} + \sqrt{1 + \left| \nabla b - \frac{1}{2} \nabla L \right|^2} \right) + \sqrt{1 + \frac{1}{4} |\nabla L|^2},$$

so that $\int_P D = \frac{|\partial K| + |\partial S_u K|}{2} \leq |\partial K|$. Thus:

$$\delta_u(K) = \frac{|\partial K| - |\partial S_u K|}{|\partial K|} \geq \left(\frac{1}{|\partial K|} \int_P \sqrt{N} \right)^2. \quad (4)$$

Next, we bound N from below. Since $\sqrt{a^2 - x^2} \leq a - \frac{x^2}{2a}$ we have

$$\begin{aligned}
Q &= \sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2 \right)^2 - \langle \nabla L, \nabla b \rangle^2} + \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2 \right) \\
&\leq 2 + \frac{1}{2} |\nabla L|^2 - \frac{\langle \nabla b, \nabla L \rangle^2}{2 + \frac{1}{2} |\nabla L|^2 + 2 |\nabla b|^2} \leq 2 + \frac{1}{2} |\nabla L|^2 \\
&\leq 2 \frac{R^2}{\rho^2} + \frac{1}{2} |\nabla L|^2 \leq \frac{4R^2}{\rho^2},
\end{aligned}$$

where the last inequality is due to Lemma 3.2 (here $\Delta L \leq 2R$). Therefore

$$\sqrt{N} \geq \frac{|\nabla b| \rho}{R}.$$

Plugging this back to (4), we may bound the surface area deficit:

$$\sqrt{\delta_u(K)} \geq \frac{1}{R |\partial K|} \int_P |\nabla b| \rho. \quad (5)$$

In order to bound the Nikodym pseudo metric by the integral of the barycenter, we first note that since $K \subseteq RD_n$, we have $|b| \leq R$. Moreover, K may be shifted parallel to u , so without loss of generality we may assume $\int_P b\rho = 0$. This shift cannot exceed R , thus the new barycenter is bounded by $2R$. We have:

$$A(K, S_u K) \leq \frac{|K \Delta S_u K|}{|K|} \leq \frac{1}{|K|} \int_P |b|, \quad (6)$$

where the second inequality is due to the fact that $||[-\frac{L}{2}, \frac{L}{2}] \Delta [b - \frac{L}{2}, b + \frac{L}{2}]| \leq |b|$. Since we assumed that $\int_P b\rho = 0$, we may apply Theorem 2.1 to get (recall $||b||_\infty \leq 2R$):

$$\begin{aligned} A(K, S_u K) &\leq \frac{1}{|K|} \left(\frac{C||b||_\infty|P|}{\beta} \right)^{1-\frac{1}{\lambda}} \left(\frac{nR}{r} \int_P |\nabla b| \rho \right)^{\frac{1}{\lambda}} \\ &\leq \frac{1}{|K|} \left(\frac{2CR|P|}{\beta} \right)^{1-\frac{1}{\lambda}} \left(\frac{nR^2|\partial K|}{r} \sqrt{\delta_u(K)} \right)^{\frac{1}{\lambda}} \\ &= \frac{|\partial K|^{\frac{1}{\lambda}}}{|K|} \left(\frac{2C|P|}{\beta} \right)^{1-\frac{1}{\lambda}} \frac{n^{\frac{1}{\lambda}} R^{1+\frac{1}{\lambda}}}{r^{\frac{1}{\lambda}}} \delta_u(K)^{\frac{1}{2\lambda}} \\ &\leq \frac{|\partial K|}{|K|} \left(\frac{C}{\beta} \right)^{1-\frac{1}{\lambda}} \frac{n^{\frac{1}{\lambda}} R^{1+\frac{1}{\lambda}}}{r^{\frac{1}{\lambda}}} \delta_u(K)^{\frac{1}{2\lambda}} \\ &\leq \left(\frac{C}{\beta} \right)^{1-\frac{1}{\lambda}} \left(n \frac{R}{r} \right)^{1+\frac{1}{\lambda}} \delta_u(K)^{\frac{1}{2\lambda}}. \end{aligned}$$

The last two inequalities hold since P is a $n - 1$ dimensional set contained in $S_u K$, thus $2|P| \leq |\partial S_u K| \leq |\partial K|$. Moreover, by Lemma 2.2, $\frac{|\partial K|}{|K|} \leq \frac{n}{r}$. \square

4 Rate of convergence

In this section we prove Theorem 1.2. The proof is based on the following idea. Assume that $|K| = |D_n|$. Due to Theorem 3.1, as long as one can find a direction u for which $A(K, R_u K)$ is not very small, there exists a Steiner symmetrization which reduces the surface area of K by a factor. Since the surface area cannot drop below $n|D_n|$ (isoperimetric inequality), the number of such operations is bounded. Next, one has to show that if $A(K, R_u K)$ is small in every direction, then so is $A(K, D_n)$. Let us formulate this last statemnt precisely before proving the main theorem.

Lemma 4.1. *Let $K \subset \mathbb{R}^n$ be a compact star shaped body and let $\varepsilon > 0$. Denote by R_u the reflection with respect to u^\perp . If $A(K, R_u K) < \varepsilon$ for all $u \in S^{n-1}$, then $A(K, D_n) < 4n\varepsilon$.*

Proof. First note that $A(K, R_{u_m} \dots R_{u_1} K) < m\varepsilon$ for any $m \leq n$. The proof goes by induction, where the case $m = 1$ is assumed to hold. For $m \geq 2$ one has

$$\begin{aligned} A(K, R_{u_m} \dots R_{u_1} K) &= A(R_{u_m} K, R_{u_{m-1}} \dots R_{u_1} K) \\ &\leq A(R_{u_m} K, K) + A(K, R_{u_{m-1}} \dots R_{u_1} K) \\ &< \varepsilon + (m-1)\varepsilon = m\varepsilon. \end{aligned}$$

Every isometry $u \in O(n)$ is generated by at most n reflections, thus $A(K, uK) < n\varepsilon$. This may be written as follows, in terms of the radial function ρ of K :

$$A(K, uK) = \frac{|K \Delta uK|}{|K|} = \frac{|D_n|}{|K|} \int_{S^{n-1}} |\rho(x)^n - \rho(ux)^n| d\sigma(x) < n\varepsilon, \quad (7)$$

where σ is the normalized Haar measure on the sphere. Without loss of generality, assume from now on that $|K| = |D_n|$. Note that if u is selected at random with respect to the Haar measure on $SO(n)$, then for every $x \in S^{n-1}$, the point ux is distributed uniformly on S^{n-1} . Thus averaging (7) over $u \in SO(n)$ yields:

$$\int_{S^{n-1}} \int_{S^{n-1}} |\rho(x)^n - \rho(y)^n| d\sigma(x) d\sigma(y) < n\varepsilon. \quad (8)$$

Consider the sets $A = \{x \in S^{n-1} : \rho(x) \geq 1\}$ and $B = \{x \in S^{n-1} : \rho(x) \leq 1\}$. Since $|K \setminus D_n| = |D_n \setminus K|$, we have

$$\begin{aligned} \frac{1}{2} A(K, D_n) &= \frac{1}{2} \int_{S^{n-1}} |\rho(x)^n - 1| d\sigma(x) = \int_A |\rho(x)^n - 1| d\sigma(x) \\ &= \frac{1}{\sigma(B)} \int_B \int_A |\rho(x)^n - 1| d\sigma(x) d\sigma(y) \\ &\leq \frac{1}{\sigma(B)} \int_B \int_A |\rho(x)^n - \rho(y)^n| d\sigma(x) d\sigma(y) \\ &\leq \frac{1}{\sigma(B)} \int_{S^{n-1}} \int_{S^{n-1}} |\rho(x)^n - \rho(y)^n| d\sigma(x) d\sigma(y). \end{aligned}$$

This implies $A(K, D_n) < \frac{2n\varepsilon}{\sigma(B)}$ by (8), and similarly one has $A(K, D_n) < \frac{2n\varepsilon}{\sigma(A)}$, so combining the two we get

$$A(K, D_n) < 4n\varepsilon.$$

□

Proof of Theorem 1.2. Assume without loss of generality that $|K| = |D_n|$. Apply n Steiner symmetrizations to K with respect to some orthogonal basis to obtain a new convex body K_0 which is unconditional, and in particular centrally symmetric. By John's theorem, there exists an ellipsoid \mathcal{E} such that

$$\mathcal{E} \subset K_0 \subset \sqrt{n}\mathcal{E}.$$

There exist n Steiner symmetrizations which transform \mathcal{E} to an Euclidean ball (see [7], Lemma 2.6). Applying these symmetrizations to K_0 , we obtain a body K_1 satisfying

$$r_1 D_n \subset K_1 \subset \sqrt{n} r_1 D_n,$$

for some $r_1 > 0$. Thus the inner and outer radii of K_1 satisfy $\frac{R}{r} \leq \sqrt{n}$. Note that $|D_n| = |K_1| \leq |\sqrt{n} r_1 D_n|$ which implies that $\frac{1}{r} \leq \sqrt{n}$. Hence, by Lemma 2.2

$$|\partial K_1| \leq \frac{n}{r} |K_1| \leq n^{3/2} |D_n|.$$

Fix $\varepsilon_0 > 0$. If there exists $u_1 \in S^{n-1}$ with $A(K_1, S_{u_1} K_1) > \varepsilon_0$, denote $K_2 = S_{u_1} K_1$. By Theorem 3.1, combined with the bound $\frac{R}{r} \leq \sqrt{n}$, we get

$$\begin{aligned} n|D_n| = |\partial D_n| \leq |\partial K_2| &\leq |\partial K_1| \left(1 - \frac{A(K_1, S_{u_1} K_1)^{2\lambda}}{n^{3(\lambda+1)}} \left(\frac{\beta}{C} \right)^{2(\lambda-1)} \right) \\ &\leq |\partial K_1| \left(1 - \frac{\varepsilon_0^{2\lambda}}{n^{3(\lambda+1)}} \left(\frac{\beta}{C} \right)^{2(\lambda-1)} \right). \end{aligned}$$

If there exists $u_2 \in S^{n-1}$ with $A(K_2, S_{u_2} K_2) > \varepsilon_0$, denote by $K_3 = S_{u_2} K_2$. Continue this process for m steps. Then K_{m+1} satisfies

$$\begin{aligned} n|D_n| \leq |\partial K_{m+1}| &\leq |\partial K_1| \left(1 - \left(\frac{\beta}{C} \right)^{2(\lambda-1)} \frac{\varepsilon_0^{2\lambda}}{n^{3(\lambda+1)}} \right)^m \\ &\leq n^{3/2} |D_n| \left(1 - \left(\frac{\beta}{C} \right)^{2(\lambda-1)} \frac{\varepsilon_0^{2\lambda}}{n^{3(\lambda+1)}} \right)^m. \end{aligned}$$

Thus,

$$0 \leq \frac{1}{2} \log n + m \log \left(1 - \left(\frac{\beta}{C} \right)^{2(\lambda-1)} \frac{\varepsilon_0^{2\lambda}}{n^{3(\lambda+1)}} \right).$$

Set $\lambda = 2 + \frac{1}{\log n}$, so $\beta = \frac{1}{1+\log n}$. Hence, the number of such steps is bounded by

$$m \leq (C(1 + \log n))^{2+\frac{2}{\log n}} \log n \left(\frac{n^{9+\frac{3}{\log n}}}{\frac{4+\frac{2}{\log n}}{\varepsilon_0}} \right) < c \left(\frac{n^9 \log^3 n}{\frac{4+\frac{2}{\log n}}{\varepsilon_0}} \right), \quad (9)$$

for some $c > 0$. The resulting body K' thus satisfies $A(K', S_u K') < \varepsilon_0$ for all $u \in S^{n-1}$, which in turn implies that $A(K', R_u K') < 2\varepsilon_0$ for all $u \in S^{n-1}$, where $R_u K_m$ is the reflection of K_m with respect to u^\perp . By Lemma 4.1 we conclude that

$$A(K_m, D_n) < 8n\varepsilon_0.$$

Let $\varepsilon > 0$. Plugging $\varepsilon_0 = \varepsilon/(8n)$ into (9) completes the proof. \square

Remark 4.2. The dependence in the dimension n in Theorem 1.2 is clearly not optimal (as mentioned before, the sharp bound is believed to be linear). For example, the bound for the ratio R/r may be reduced to a constant, rather than \sqrt{n} , which results in decreasing the power 13 to 10. This may be done by one of the isomorphic results mentioned in the introduction.

References

- [1] Barchiesi M., Cagnetti F., Fusco N., *Stability of the Steiner Symmetrization of Convex Sets*. Journal of the European Mathematical Society, 15 (4). 1245–1278 (2013)
- [2] Bourgain J., Lindenstrauss J., Milman V.D., *Estimates Related to Steiner Symmetrizations*. Geometric Aspects of Functional Analysis (1987-88). Lect. Notes Math., Vol. 1376, 264-273 (1989).
- [3] Chua S., Wheeden R., *Estimates of Best Constants for Weighted Poincaré Inequalities on Convex Domains*, Proceedings of LMS 93, 197–226 (2006).
- [4] Gross W., *Die Minimaleigenschaft der Kugel*. Monatsh. Math. Phys. 28, no. 1, 77–97 (1917).
- [5] Hadwiger H., *Einfache Herleitung der Isoperimetrischen Ungleichung für Abgeschlossene Punktmengen*. Math. Ann. 124, 158-160 (1952).
- [6] Klartag B., *$5n$ Minkowski Symmetrizations Suffice to Arrive at an Approximate Euclidean Ball*. Annals of Math., Vol. 156, No. 3, 947-960 (2002).
- [7] Klartag B., Milman V.D., *Isomorphic Steiner Symmetrizations*. Invent. Math., Vol. 153, No. 3, 463-485 (2003).
- [8] Klartag B., *Rate of Convergence of Geometric Symmetrization*. Geom. and Funct. Anal. (GAFA), Vol. 14, Issue 6, 1322–1338 (2004).
- [9] Mani-Levitska P., *Random Steiner Symmetrizations*. Studia Sci. Math. Hungar. 21, no. 3–4, 373–378 (1986).

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